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LATERAL BUCKLING OF CANTILEVERED I-BEAMS UNDER UNIFORM LOADS

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LATERAL BUCKLING OF CANTILEVERED I-BEAMS UNDER UNIFORM LOADS

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INTRODUCTION

In recent years, the engineer has become progressively more conscious of the weight to rigidity ratio of structures. This has led him to reexamine questions of design of structural units from a more exact standpoint. As a result, he has encountered boundary value problems in elastic stability which are intractable by analytical means. Before the advent of the modern high speed computing machine, the labor involved in obtaining numerical solutions made numerical methods impractical in many important cases. An example of one of these is the problem of lateral buckling of a cantilevered I-beam subjected to a uniform load. A numerical solution of this problem, obtained with the aid of the IBM Card Programmed Calculator, is presented herein.

Differential Equations and Boundary Conditions

With coordinate axes as shown in Figure 1, the total energy of an I-beam buckled laterally under the action of a uniform normal load acting along the centroidal axis of the cross section is given by (1)

$$V=\frac{1}{2}\int_0^L\left[B(u'')^2+C(\beta')^2+\frac{Dh^2}{2}(\beta'')^2+2M\beta u''\right]dz \qquad (1)$$ [2]

where

B = minimum flexural rigidity of beam.
u = lateral deflection of beam in plane of minimum rigidity.
L = length of beam.
D = flexural rigidity of one flange in its own plane.
h = depth of beam.
β = angular rotation of section of abscissa z.
C = torsional rigidity of cross-section.
M = $\frac{q}{2}(L-z)^2$ = bending moment in the plane of maximum flexural rigidity of section of abscissa z.
q = intensity of uniform normal load.

Eq. (1) is derived under the same assumptions used by Timoshenko (2) in his "Theory of Elastic Stability", Chapter V; it applies therefore to doubly symmetrical I-sections with the principal moment of inertia I y much smaller than I x.

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2. Primes indicate differentiation with respect to z.

Applying the usual methods of the calculus of variations to Eq. (1), we may derive the differential equation governing β,

$$\beta^{IV} - \frac{2C}{Dh^2}\beta'' - \frac{2M^2}{BDh^2}\beta = 0 \tag{2}$$

and the boundary conditions

$$\beta(0) = \beta'(0) = \beta''(L) = \left(\beta''' - \frac{2C}{Dh^2}\beta'\right)_{z=L} = 0 \tag{3}$$

Letting

$$x = \frac{z}{L}$$

$$\frac{1}{a^2} = \frac{2C}{Dh^2} \tag{4a}$$

$$\frac{1}{d^4} = \frac{q^2 L^4}{2BDh^2} \tag{4b}$$

in Eq. (2), the buckling equation becomes:[3]

$$\beta^{IV} - \frac{L^2}{a^2}\beta'' - \frac{L^4}{d^4}(1-x)^4\beta = 0 \tag{5}$$

while the boundary conditions (3) are:

$$\beta(0) = \beta'(0) = \beta''(1) = \left(\beta''' - \frac{L^2}{a^2}\beta'\right)_{x=1} = 0 \tag{6}$$

Solution by Finite Differences

Subdividing the interval [0,1] into n equal parts of length $\bar{h} = 1/n$ and taking the first terms in the expansions of the derivatives in terms of central differences (3), Eq. (5) becomes:

$$\delta^4\beta_i - \frac{L^2}{a^2}\bar{h}^2\delta^2\beta_i - \frac{L^4}{d^4}\bar{h}^4(1-x_i)^4\beta_i = 0 \tag{7}$$

where $\beta_i \equiv \beta(x_i)$, x_i being a pivotal abscissa of the interval [0,1]. Substituting in Eq. (7) the values of the central differences in terms of pivotal values, we obtain:

$$\beta_{i-2} - S\beta_{i-1} + \left[2(S-1) - K^2(1-x_i)^4\right]\beta_i - S\beta_{i+1} + \beta_{i+2} = 0 \tag{8}$$

3. From here on, primes indicate differentiation with respect to x.

where

$$S = 4 + \left(\frac{L}{a}\bar{h}\right)^2 \tag{9a}$$

$$K^2 = \left(\frac{L}{d}\bar{h}\right)^4 \tag{9b}$$

Similarly, the boundary conditions (8) become:

$$\begin{aligned} &\beta_0 = 0 \\ &\beta_{-1} = \beta_1 \\ &\beta_{n+1} = 2\beta_n - \beta_{n-1} \\ &\beta_{n+2} = 2(S-2)(\beta_n - \beta_{n-1}) + \beta_{n-2} \end{aligned} \tag{10}$$

From equations (4a), (4b) and (9b), the critical total load takes the form

$$(qL)_{cr.} = m \frac{\sqrt{BC}}{L^2} \tag{11}$$

where

$$m = \frac{2Kn^2}{(L/a)} \tag{12}$$

To solve the boundary value problem defined by Eq's. (8), (10) one fixes a value of L^2/a^2, and a value of n. Eq. (8) is then applied at the pivotal x_i $(i=1,2,\cdots,n)$ using Eq. (10) to eliminate β_0, β_{-1}, β_{n+1}, β_{n+2}. This yields a set of n linear, algebraic, homogeneous equations in the n unknowns $\beta_1, \cdots, \beta_n$ and this system has solutions β_i not identically zero, if and only if its determinant Δ, which is a function of K^2, is equal to zero. In problems of elastic stability, the designer will be mainly interested in the lowest buckling load, and hence the determinental equation

$$\Delta(K^2) = 0 \tag{13}$$

is solved for its smallest root. Having done this for one value of n, the operation is repeated with a larger n to obtain a better approximation to the true value of K^2. By increase in the number n of pivotal points, K^2 may be evaluated, theoretically, to any degree of accuracy.

For large values of n, the evaluation of the roots of $\Delta(K^2) = 0$ is impractical without the use of special computing devices. The actual calculations for this problem were performed on the I.B.M. Card Programmed Electronic Calculator, Model II, which was made available to

the author by the Watson Scientific Computing Laboratory, at Columbia University. A detailed account of the method employed for these calculations is to be found else where [4].

Results

The following table gives the factor m versus L^2/a^2. For a given I-beam, L^2/a^2 is computed by means of the formula

$$\frac{L^2}{a^2} = \frac{2CL^2}{Dh^2} \tag{14}$$

The corresponding value of m is obtained from the table, and the critical total load is calculated from Eq. (11).

L^2/a^2	1	2	3	4	6	8	10
m	66.9	50.9	44.0	39.8	34.9	31.9	29.8
L^2/a^2	12	14	16	24	32	40	100
m	28.3	27.1	26.0	23.5	21.9	20.9	17.6

We see that, as the ratio L^2/a^2 increases, the factor m diminishes and approaches as a limit the value m = 12.85, obtained for lateral buckling of a cantilivered rectangular beam under similar loading conditions [5].

Example

Consider a 24I79.9 section, with L = 20 ft. The flexural rigidity D of a flange can be taken equal to half of the flexural rigidity of the I-beam in the lateral direction, while the torsional rigidity C may be calculated from the approximate formula [6],

$$C = \frac{G}{3}(2bt^3 + ht_1^3)$$

where b is the flange width, t the average flange thickness, h the beam depth, t_1 the web thickness, and G the shear modulus. For the above section, b = 7", t = 0.87", t_1 = 0.5", h = 24", B = 42.9E in^4 and D = 21.5E in^4, E being the elastic modulus. Taking $E = 2.6G$, we get C = 1.57 in^4. Thus

$$\frac{L^2}{a^2} = \frac{2CL^2}{Dh^2} = \frac{2 \times 1.57 \times 20^2}{21.5 \times 2^2} = 14.6$$

From the table, we obtain m = 26.8. With $E = 30 \times 10^6$ psi, we have

$$(qL)_{cr.} = m\frac{\sqrt{BC}}{L^2} = 26.8 \times 30 \times 10^6 \times \frac{\sqrt{42.9 \times 1.57}}{(20 \times 12)^2} \;{}^{\#}$$

$$= 115{,}000^{\#}$$

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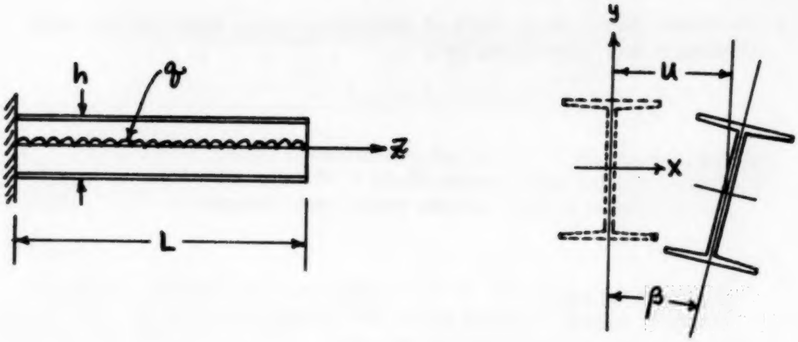

Figure 1 - Coordinate Axes